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Asymptotic distribution of zeros of polynomials satisfying difference equations

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Abstract

We propose a way to find the asymptotic distribution of zeros of orthogonal polynomials $p_n(x)$ satisfying a difference equation of the form

$$B(x)p_n(x+\delta) - C(x,n)p_n(x) + D(x)p_n(x-\delta) = 0.$$

We calculate the asymptotic distribution of zeros and asymptotics of extreme zeros of the Meixner and Meixner–Pollaczek polynomials. The distribution of zeros of Meixner polynomials shows some delicate features. We indicate the relation of our technique to the approach based on the Nevai–Dehesa–Ullman distribution.

Keywords: Orthogonal polynomials; Difference equations

1. Introduction

The hypergeometric orthogonal polynomials $p_n(x)$ that appear in the Askey scheme [9] satisfy either differential equations or difference equations of the form

$$B(x)p_n(x+\delta) - C(x,n)p_n(x) + D(x)p_n(x-\delta) = 0, \tag{1}$$

where δ is equal to 1 or i when the polynomials are orthogonal with a discrete or continuous weight $\alpha(x)$, respectively.

As is well known (see, e.g., [6]), the zeros of $p_n(x)$ are real and simple. In the present paper, we will propose a procedure to calculate the asymptotic distribution of the zeros of $p_n(nz)$ as $n \to \infty$

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by applying a form of Bethe ansatz to Eq. (1).[1] The difference between the cases $\delta = \mathrm{i}$ and 1 will be clearly visible.

Note that although we focus on the orthogonal polynomials belonging to the Askey scheme, our results have a more general application. In fact, we need only the two conditions: (1) polynomials $p_n(x)$ satisfy the difference equation (1) (or a certain type of q-difference equation [10]); (2) the density of zeros is piece-wise smooth on a one-dimensional manifold. For the polynomials in the Askey scheme, the latter manifold is just the real line. For another example, where the manifold is the unit circle, see [10].

In this work, we are interested in zeros of $p_n(x)$ only to the main order in n. However, the method we apply should remain useful for extraction of the further terms in the asymptotic series.

Other approaches to the calculation of the asymptotic distribution of zeros of orthogonal polynomials are based on the knowledge of the recurrence relation, or the weight function $\mathrm{d}\alpha(x)/\mathrm{d}x$, or the asymptotic behaviour of polynomials themselves (see [12] for a review). Our approach is closely related to the semiclassical (WKB) analysis of differential (difference) equations.

We will consider in detail the cases of the Meixner polynomials (orthogonal on the set $\{0, 1, 2, \dots\}$, $\delta = 1$) and the Meixner–Pollaczek polynomials (orthogonal on the set $(-\infty, \infty)$, $\delta = \mathrm{i}$). The asymptotics of zeros of these polynomials was investigated in [5,8,17], by other methods. Therefore, the reader can compare several approaches. Our main purpose is to demonstrate how Bethe ansatz works in this context, and we rederive some known facts. We formulate them, for convenience, as theorems. However, part of our results on the Meixner polynomials are new.

The zeros of $p_n(x)$ are intimately related to the points of increase of the weight (see, e.g., [6]). If the moment problem corresponding to the polynomials $p_n(x)$ is determinate, the points of increase of the weight coincide with the closure of the set of zeros of $p_n(x)$, $n \to \infty$. For the Meixner and the Meixner–Pollaczek polynomials the latter fact implies that the density $\rho(z)$ of the zeros of $p_n(nz)$ at the point $z = 0$ is 1 and ∞, respectively. Naturally, the expressions for $\rho(z)$ we obtain have this property. It is interesting that, moreover, for the Meixner polynomials $\rho(z) = 1$ in the *finite* vicinity of $z = 0$. In this vicinity the distribution of zeros is characterized by another function (simply related to $\rho(z)$) that gives the rate with which the distance between zeros approaches the value 1 as $z \to 0$. This rate is exponential in n.

There is another interpretation of our results on the Meixner and Meixner–Pollaczek polynomials. Consider a symmetric tridiagonal (Jacobi) matrix with real matrix elements ($J_{ik} = J_{ki}$, $i, k = 0, 1, 2, \dots$, $J_{ik} = 0$ if $|i - k| > 1$) such that the limits $\lim_{n\to\infty} J_{nn}/\varphi(n) = a$, $\lim_{n\to\infty} J_{n\ n+1}/\varphi(n) = b/2$ are finite and not simultaneously zero and $\varphi(n)$ is a nondecreasing function. All such matrices for $\varphi(n) = n$ are separated in two classes. The first class is comprised of matrices related to the Meixner polynomials and the second, to Meixner–Pollaczek. The asymptotics of zeros of polynomials is the asymptotics of eigenvalues of the corresponding truncated Jacobi matrices. Thus in this paper we have calculated, in particular, the asymptotic distribution of eigenvalues of the above described Jacobi matrices for $\varphi(n) = n$. On the other hand, the density of zeros in the case of $a = 0$ is known for any $\varphi(n)$ (with some limitations on the form of $\varphi(n)$). It is called Nevai–Dehesa–Ullman distribution [13,15]. It can be easily generalized to the case of any real a, b. In particular for $\varphi(n) = n$, this gives an alternative derivation of the density of zeros of the Meixner and Meixner–Pollaczek

[1] We will consider infinite systems of orthogonal polynomials.

polynomials. However, this does not reproduce the delicate features of the zero distribution of the Meixner polynomials in the vicinity of $z=0$.

The plan of this work is the following. In Section 1 we introduce the Bethe ansatz technique in the simplest case of polynomials satisfying differential equations. We obtain the asymptotic density of zeros and compare the results with those of a more straightforward WKB-type approach.

In Section 2 we consider the difference equation (1) with $\delta = \mathrm{i}$. It is satisfied by polynomials orthogonal with a continuous weight: Meixner–Pollaczek, continuous Hahn, continuous dual Hahn, Wilson. After getting the general solution, we focus on Meixner–Pollaczek as the other polynomials have, to the main order in n, the density of zeros simpler than that of Meixner–Pollaczek.

In Section 3 we consider the more interesting case of Eq. (1) with $\delta=1$. Such equation is satisfied by the Meixner and Charlier polynomials. We will consider only them, as other polynomials with discrete weights in the Askey scheme are finite systems.

In Section 4 we consider the asymptotic eigenvalue problem for symmetric tridiagonal matrices.

2. Differential equation

We shall start with the case of a differential equation as it is the simplest one, and it already shows some features of the method.

Consider a system of polynomials satisfying the following differential equation: [2]

$$a(x)y_n'' + b(x)y_n' + c(x,n)y_n = 0. \tag{2}$$

Here $y_n(x)$ is a polynomial of exactly degree n, functions $a(x)$, $b(x)$, $c(x,n)$, where only the last one depends on n, are smooth with possible singular points that do not coincide with any zeros of $y_n(x)$. We shall assume that there is such a constant $\mu>0$ that $c(x,n)\sim c_\infty(x)n^{2\mu}$ as $n\to\infty$. Such equations are satisfied, e.g., by the Jacobi, Laguerre, Hermite polynomials. These polynomials are orthogonal with an absolutely continuous weight on an interval $\mathscr{A}$ (unbounded in the case of Laguerre and Hermite) and the zeros of $y_n(x)$ become everywhere dense in $\mathscr{A}$ as $n\to\infty$.

Let $x_1<x_2<\cdots<x_n$ be the zeros of $y_n(x)$. Up to a constant factor, we can write $y_n(x)$ in the form $y_n(x)=\prod_{k=1}^n(x-x_k)$. Substituting this expression in (2) and dividing the equation by $y_n(x)$, we get

$$a(x)\sum_{k=1}^{n}\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{(x-x_k)(x-x_i)} + b(x)\sum_{k=1}^{n}\frac{1}{x-x_k} + c(x,n) = 0. \tag{3}$$

Obviously, the singularities of the function at the left-hand side must vanish. Equating the residues in the simple poles x_k, $k=1,\dots,n$, to zero, we obtain the following system:

$$2a(x_k)\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x_k-x_i} + b(x_k) = 0, \quad k=1,\dots,n. \tag{4}$$

[2] Here we restrict our attention to second-order differential equations, but it is easy to generalize our approach to higher-order equations.

Eq. (4) can also be obtained by dividing (2) by $y_n'(x)$ and putting $x = x_k$. The quantities $\sigma_{pk} = \sum_{i=1, i\neq k}^{n} 1/(x_k - x_i)^p$, $p = 1, 2, \ldots$, and related ones were calculated in [1,2,4]. Analogous algebraic relations for solutions of some q-difference equations are given in [18]. Because of the analogy to the theory of integrable models, the equations of type (4) are called Bethe ansatz equations. For our purposes, it will be sufficient to know only the asymptotics of σ_{1k} and σ_{2k} as $n \to \infty$.

Let us now differentiate (2) with respect to x and apply the same polynomial-ansatz technique to the resulting equation. Then instead of (4) we get

$$3a(x_k)\left\{\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x_k-x_i}\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{x_k-x_j}-\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{(x_k-x_i)^2}\right\}$$

$$+2(a'(x_k)+b(x_k))\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x_k-x_i}+c(x_k,n)+b'(x_k)=0, \quad k=1,\ldots,n. \tag{5}$$

Now divide (5) by $n^{2\mu}$ and take the limit as $n \to \infty$ on condition that the point $x = \lim_{n\to\infty} x_{k(n)}$, $x \in \mathscr{A}$, is fixed. Then, using the fact that, as follows from (4):

$$\lim_{n\to\infty}\frac{1}{n^{\sigma}}\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x_k-x_i}=0 \tag{6}$$

for any $\sigma > 0$, we obtain from (5)

$$\lim_{n\to\infty}\frac{1}{n^{2\mu}}\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{(x_k-x_i)^2}=\frac{c_\infty(x)}{3a(x)}. \tag{7}$$

Let us assume that the zeros are asymptotically spaced as $1/n^{\mu}$ and their asymptotic density is a piece-wise smooth function. Hence for $|i-k| \leqslant M$ we have

$$x_i - x_k = (i-k)\gamma_k/n^{\mu} \tag{8}$$

up to at most $O((M/n^{\mu})^2)$ as $M, n \to \infty$ in such a way that $M/n^{\mu} \to 0$. For brevity, we shall denote the limit $M, n \to \infty$, $M/n^{\mu} \to 0$ just by lim. In (8) $\lim \gamma_{k(n)} = \gamma(x) > 0$. The asymptotic density of zeros is, obviously, $\tilde{\rho}(x) = \lim 1/\gamma_{k(n)}$. Using (8) we rewrite the l.h.s. of (7) in the form

$$\lim\frac{1}{n^{2\mu}}\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{(x_k-x_i)^2}=2\tilde{\rho}(x)^2\sum_{i=1}^{\infty}\frac{1}{i^2}=\frac{\pi^2}{3}\,\tilde{\rho}(x)^2. \tag{9}$$

So putting (7) and (9) together, we have

$$\tilde{\rho}(x)=\frac{1}{\pi}\sqrt{\frac{c_\infty(x)}{a(x)}} \tag{10}$$

on the set $\mathscr{A}$. It is not necessary to justify (10) further as this result is, of course, well known. It is related to the semiclassical approximation in the theory of Schrödinger equation (see, e.g., [11]).

In a few words, the usual way to obtain (10) from (2) is the following. Consider a small σ-neighbourhood of a point x. If $a(x)c_\infty(x) > 0$ then, starting with a sufficiently large number n, the characteristic values of Eq. (2) (considered as an equation with constant coefficients: $a(x)$, $b(x)$, $c(x,n)$ being sufficiently smooth), $\omega_\pm = \{-b(x) \pm \sqrt{b(x)^2 - 4a(x)c(x,n)}\}/(2a(x))$, have the imaginary part growing as n^μ. Hence, it is easy to conclude that the asymptotic distance between consecutive zeros is $x_{k+1} - x_k \sim \gamma_k/n^\mu$, $\gamma_k = \pi\sqrt{a(x)/c_\infty(x)} + o(1)$.

For the polynomials orthogonal on an unbounded interval, another type of the density of zeros can be introduced. Take, for example, the Hermite polynomials. Their zeros are symmetric with respect to the origin, and the largest (smallest) zero $x_n \sim \sqrt{2n}$ ($x_1 \sim -\sqrt{2n}$). Eq. (4) in this case has the form

$$\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{x_k - x_i} = x_k, \quad k = 1, \ldots, n. \tag{11}$$

Changing the variable $x_i = z_i\sqrt{2n}$ and then taking the limit $n \to \infty$, we get [3]

$$\frac{1}{2}\,\text{V.p.} \int_{-1}^{1} \frac{\rho(\omega)\,\mathrm{d}\omega}{z - \omega} = z, \quad z \in (-1, 1), \tag{12}$$

where the density of zeros $\rho(\omega) = \lim_{n\to\infty} 1/(z_{i+1} - z_i)n$, $\omega = \lim_{n\to\infty} z_{i(n)}$. Solving (12) and using the normalization condition $\int_{-1}^{1} \rho(\omega)\,\mathrm{d}\omega = 1$, we get

$$\rho(z) = \frac{2}{\pi}\sqrt{1 - z^2}. \tag{13}$$

This way to derive (13) was proposed in [3]. Of course, there are other ways to find $\rho(z)$. Expression (13) is valid for a contracted to $[-1, 1]$ density of zeros of a large class of orthogonal polynomials (to which Hermite polynomials belong). It is a particular case of the Nevai–Dehesa–Ullman distribution (see Section 5) for $\varphi(n) = \sqrt{n}$.

Note that (10) for Hermite polynomials gives $\tilde{\rho}(x) = \sqrt{2}/\pi$.

In Section 3 we shall see how the technique we used to obtain (10) generalizes to the difference equations.

3. Difference equation—continuous weight

Consider polynomials in the Askey scheme orthogonal with a continuous weight on an unbounded interval and satisfying (1). In this case $\delta = \mathrm{i}$. Take the nth polynomial and make the change of variable $x = zn$. Then $p_n(x) = \prod_{k=1}^{n}(x - x_k) = n^n y(z)$, where $y(z) = \prod_{k=1}^{n}(z - z_k)$. As before, we assume $x_1 < x_2 < \cdots < x_n$. If $C(x,n) \sim c(z)n^\mu$ as $n \to \infty$, then set $b(z) = \lim_{n\to\infty} B(zn)/n^\mu$, $d(z) = \lim_{n\to\infty} D(zn)/n^\mu$. Thus, to the main order in n, (1) can be written in the form

$$b(z)y(z + i/n) - c(z)y(z) + d(z)y(z - i/n) = 0. \tag{14}$$

Substituting here $y(z)=\prod_{k=1}^{n}(z-z_k)$ and evaluating the equation at the zeros z_m, $m=1,2,\dots,n$, we get

$$\prod_{k=1}^{n}\frac{z_m-z_k+i/n}{z_m-z_k-i/n}=-\frac{d(z_m)}{b(z_m)},\quad m=1,2,\dots,n. \tag{15}$$

Now assume that the distance between zeros x_k is of order 1 (which is natural to expect from (1)), and the asymptotic density $\rho(z_m)=1/(x_m-x_{m-1})$ of zeros becomes a piece-wise smooth function of $z=\lim_{n\to\infty}z_{m(n)}$ as $n\to\infty$. Hence for $|p|<M$

$$z_m-z_{m-p}=\frac{p}{\rho(z)n} \tag{16}$$

up to at most $O(M^2/n^2)$ as $M,n,n/M\to\infty$. As before, we denote this limit by "lim". Applying "lim" to (15) we have

$$\lim\left(-\prod_{p=1}^{M}\frac{\frac{p}{\rho(z)}+i}{\frac{p}{\rho(z)}-i}\,\frac{\frac{-p}{\rho(z)}+i}{\frac{-p}{\rho(z)}-i}\prod_{k}{}'\,\frac{1+\frac{i/n}{z_m-z_k}}{1-\frac{i/n}{z_m-z_k}}\right)=-\frac{d(z)}{b(z)}, \tag{17}$$

where the prime indicates that the product is taken over k outside the range $m-M,\dots,m+M$. The product over p gives 1 in the limit. Since $i/n(z_m-z_k)$ in the second product is small (of order $1/M$ or less), we can perform the small-parameter expansion in it. Replacing the second product by the exponent of its logarithm we get $\lim\exp[(2\mathrm{i}/n)\sum_k'1/(z_m-z_k)]=d(z)/b(z)$ which, in turn, gives

$$\exp\left(-2\mathrm{i}\,\mathrm{V.p.}\int_{-\infty}^{\infty}\frac{\rho(\omega)\,\mathrm{d}\omega}{\omega-z}\right)=\frac{d(z)}{b(z)}. \tag{18}$$

Note that $\rho(z)$ also satisfies the normalization condition

$$\int_{-\infty}^{\infty}\rho(z)\,\mathrm{d}z=1. \tag{19}$$

Instead of trying to solve (18) directly, we will do the following. Replace z by $z+i/n$ in (14) and consider this new equation (cf. Section 2). Then we obtain instead of (15):

$$\prod_{k=1}^{n}\frac{z_m-z_k+2i/n}{z_m-z_k+i/n}=\frac{c(z_m)}{b(z_m)},\quad m=1,2,\dots,n. \tag{20}$$

Proceeding as before, we get for the l.h.s. of (20):

$$\begin{aligned}\lim\prod_{k=1}^{n}\frac{z_m-z_k+2i/n}{z_m-z_k+i/n}&=2\lim\prod_{s=0}^{[M/2]}\left\{1+\left(\frac{2\rho(z)}{2s+1}\right)^2\right\}\exp\left(\frac{\mathrm{i}}{n}\sum_k{}'\frac{1}{z_m-z_k}\right)\\&=2\cosh(\pi\rho(z))\exp\left(-\mathrm{i}\,\mathrm{V.p.}\int_{-\infty}^{\infty}\frac{\rho(\omega)\,\mathrm{d}\omega}{\omega-z}\right).\end{aligned} \tag{21}$$

Thus, in the limit equation (20) becomes

$$2\cosh(\pi\rho(z))\exp\left(-\mathrm{i}\,\mathrm{V.p.}\int_{-\infty}^{\infty}\frac{\rho(\omega)\,\mathrm{d}\omega}{\omega-z}\right)=\frac{c(z)}{b(z)}. \tag{22}$$

Putting (18) and (22) together, we obtain

$$\rho(z)=\frac{1}{\pi}\operatorname{arccosh}\left|\frac{c(z)}{2\sqrt{b(z)d(z)}}\right|. \tag{23}$$

Thus $\rho(z)$ can be either (23) or zero. If we suppose that $\rho(z)$ is defined by (23) for all z where $|c(z)/(2\sqrt{b(z)d(z)})|>1$ (hence $\rho(z)=0$ otherwise) then the asymptotic values of the largest and smallest zeros are obtained from the equations $c(z)/(2\sqrt{b(z)d(z)})=\pm 1$.

In the concrete cases we can verify if thus obtained $\rho(z)$ is indeed the asymptotic density of zeros by substituting it in (18), (19), (22) and integrating. These concrete cases include the Wilson, continuous Hahn, continuous dual Hahn, and Meixner–Pollaczek polynomials [9].

The most interesting example is provided by the Meixner–Pollaczek polynomials. They are orthogonal on the line $(-\infty,\infty)$ with the weight function $\mathrm{d}\alpha(x)/\mathrm{d}x=\exp[(2\phi-\pi)x]|\Gamma(\lambda+\mathrm{i}x)|^2$, where $\lambda>0$, $0<\phi<\pi$. (Note that $\mathrm{d}\alpha(x)$ can be obtained [14] from the difference equation). The coefficients in Eq. (1) for these polynomials are the following: $B(x)=\mathrm{e}^{\mathrm{i}\phi}(\lambda-\mathrm{i}x)$, $C(x,n)=2\mathrm{i}[(n+\lambda)\sin\phi-x\cos\phi]$, $D(x)=-\mathrm{e}^{-\mathrm{i}\phi}(\lambda+\mathrm{i}x)$, and $\delta=\mathrm{i}$.

Using the just obtained general results, we can immediately formulate a theorem.

Theorem 1. *For the Meixner–Pollaczek polynomials $p_n(zn)$ the asymptotic density of zeros is*

$$\rho(z)=\frac{1}{\pi}\operatorname{arccosh}\left|\frac{\sin\phi}{z}-\cos\phi\right|,\quad \textit{if } z\in\left[-\cot\frac{\phi}{2},\tan\frac{\phi}{2}\right],$$

and $\rho(z)=0$ otherwise.

We note that the values $n\tan\phi/2$ and $-n\cot\phi/2$ are the asymptotics of the largest and smallest zeros of the Meixner–Pollaczek polynomials $p_n(x)$, respectively. There can be no isolated zeros z_m outside the interval $[-\cot\phi/2,\tan\phi/2]$ as the Bethe ansatz equations (15) and (20) would not hold for such zeros. We can also use here another argument. Consider the Gerschgorin circles of the Jacobi matrix associated with $p_n(x)$ (see Section 5). We then find that all zeros must be inside the interval $[-\cot\phi/2,\tan\phi/2]$.

On the assumption of smoothness of $\rho(z)$, it is easy to prove the theorem by verifying (18), (19), and (22). In particular, we have

$$\mathrm{V.p.}\int_{-\cot\phi/2}^{\tan\phi/2}\frac{\rho(\omega)\,\mathrm{d}\omega}{\omega-z}=\begin{cases}\phi-\pi, & 0<z<\tan\phi/2,\\ \phi, & -\cot\phi/2<z<0.\end{cases} \tag{24}$$

As such, our results for the Meixner–Pollaczek polynomials are not new. In the implicit form they are contained in [5,17].

In Section 4 we will use Bethe ansatz techniques to indicate some interesting features of the distribution of zeros of Meixner polynomials in the region where the asymptotic density of zeros is constant.

4. Difference equation—discrete weight

The Meixner and Charlier polynomials orthogonal on the set $\{0,1,2,\ldots\}$ satisfy Eq. (1) with $\delta=1$. Using the same notation as in Section 3, we reduce (1) to the form

$$b(z)y(z+1/n)-c(z)y(z)+d(z)y(z-1/n)=0. \tag{25}$$

Evaluating it at the zeros z_m, $m=1,2,\ldots,n$, we get

$$\prod_{k=1}^{n}\frac{z_m-z_k+1/n}{z_m-z_k-1/n}=-\frac{d(z_m)}{b(z_m)},\quad m=1,2,\ldots,n. \tag{26}$$

For the distance between nearest zeros we still have the expression (16). However, as we can notice from (26), there appears a new important feature. We mentioned in the introduction that in the present case $\rho(z)\to 1$ as $z\to 0$. Hence we can expect that $z_m-z_{m-1}-1/n=o(1/n)$ and $z_m-z_{m+1}+1/n=o(1/n)$ for z_m close to zero. Therefore, to the main order in n, the ratio of these quantities is an unknown if $\rho(z_m)=1$ and equal to -1 if $\rho(z_m)\neq 1$.[3] We denote $\Delta_{m-1}=z_m-z_{m-1}-1/n$. Now we can proceed as in the previous section

$$\begin{aligned}\lim\prod_{k=1}^{n}\frac{z_m-z_k+1/n}{z_m-z_k-1/n}&=\lim\left(-\frac{z_m-z_{m+1}+1/n}{z_m-z_{m+1}-1/n}\,\frac{z_m-z_{m-1}+1/n}{z_m-z_{m-1}-1/n}\right.\\&\qquad\left.\times\prod_{p=2}^{M}\frac{\frac{p}{\rho(z)}+1}{\frac{p}{\rho(z)}-1}\,\frac{\frac{-p}{\rho(z)}+1}{\frac{-p}{\rho(z)}-1}\prod_k{}'\,\frac{1+\frac{1/n}{z_m-z_k}}{1-\frac{1/n}{z_m-z_k}}\right)\\&=-\lim\left(\frac{\Delta_m}{\Delta_{m-1}}\exp\left[\frac{2}{n}\sum_k{}'\frac{1}{z_m-z_k}\right]\right)=-\frac{d(z)}{b(z)}.\end{aligned} \tag{27}$$

Thus, in the limit (26) becomes

$$\chi(z)\exp\left(-2\,\mathrm{V.p.}\int_{-\infty}^{\infty}\frac{\rho(\omega)\,\mathrm{d}\omega}{\omega-z}\right)=\frac{d(z)}{b(z)}, \tag{28}$$

where $\chi(z)=\lim(\Delta_m/\Delta_{m-1})$, $z=\lim_{n\to\infty}z_{m(n)}$.

Remember that $\rho(z)$ satisfies the normalization condition

$$\int_{-\infty}^{\infty}\rho(z)\,\mathrm{d}z=1. \tag{29}$$

As in Section 3, change the variable $z\to z+1/n$ in (25). Evaluating thus obtained equation at the zeros z_m, we have

$$\prod_{k=1}^{n}\frac{z_m-z_k+2/n}{z_m-z_k+1/n}=\frac{c(z_m)}{b(z_m)},\quad m=1,2,\ldots,n. \tag{30}$$

[3] Strictly speaking, we also had to consider the cases when $\rho(z_m)=s$, where s is an integer greater that 1. In this case $z_m-z_{m\pm s}\pm 1/n=o(1/n)$. However, we shall see that $\rho(z)\leqslant 1$ for all z *at least* for the Meixner and Charlier polynomials.

In the limit (30) becomes

$$\lim\left\{\left(1+\frac{\Delta_{m+1}}{\Delta_m}\right)\prod_{s=0}^{[M/2]}\left[1-\left(\frac{2\rho(z)}{2s+1}\right)^2\right]\exp\left(\frac{1}{n}\sum_k{}'\frac{1}{z_m-z_k}\right)\right\}=\frac{c(z)}{b(z)}. \tag{31}$$

Since $\lim(\Delta_{m+1}/\Delta_m)=\chi(z)$, we may rewrite (31) in the form

$$(1+\chi(z))\cos(\pi\rho(z))\exp\left(-\text{V.p.}\int_{-\infty}^{\infty}\frac{\rho(\omega)\,\mathrm{d}\omega}{\omega-z}\right)=\frac{c(z)}{b(z)}. \tag{32}$$

From (28) and (32) we obtain a simple relation between $\rho(z)$ and $\chi(z)$:

$$(1+\chi(z))\cos(\pi\rho(z))=\frac{c(z)}{b(z)}\sqrt{\frac{\chi(z)b(z)}{d(z)}}. \tag{33}$$

We know that $\chi(z)=1$ whenever $\rho(z)$ is nonconstant. Hence, if $\rho(z)$ is nonconstant, it satisfies the equation

$$\cos(\pi\rho(z))=\frac{c(z)}{2b(z)}\sqrt{\frac{b(z)}{d(z)}}. \tag{34}$$

Obviously, this equation can hold only in the region where the absolute value of the r.h.s. is no more than 1. Having guessed $\rho(z)$ and $\chi(z)$ from (33), we can check if we are right by verifying (28), (32), and (29). Thus, we get the following theorem for the Meixner polynomials $p_n(x)$ for which [9] $B(x)=(x+\beta)c$, $C(x)=x+(x+\beta)c+(c-1)n$, $D(x)=x$, $\beta>0$, $0<c<1$.

Theorem 2. *For the Meixner polynomials $p_n(zn)$ the asymptotic density of zeros is*

$$\rho(z)=\begin{cases}1, & z\in[0,\alpha^{-1}],\\ \dfrac{1}{\pi}\arccos f(z), & z\in[\alpha^{-1},\alpha],\\ 0, & z\notin[0,\alpha],\end{cases}$$

where $0\leqslant\arccos z\leqslant\pi$, and

$$\alpha=\frac{1+\sqrt{c}}{1-\sqrt{c}},\qquad f(z)=\frac{(c+1)z+c-1}{2\sqrt{cz}}.$$

The function

$$\ln\chi(z)=\begin{cases}2\,\mathrm{arccosh}\,|f(z)|, & z\in[0,\alpha^{-1}],\\ 0, & z\in[\alpha^{-1},\alpha].\end{cases}$$

Remark. Since in the limit $\Delta_{m-1}=\exp[-(k/n)n\ln\chi(z)]\Delta_{m-1+k}$, and $\chi(z)>1$ for $z\in[0,\alpha^{-1}]$, the function $\ln\chi(z)$ describes the exponential rate with which zeros x_k approach integer values as z decreases from the point α^{-1}.

The function $\rho(z)$ is continuous on the set $(0,\infty)$. Its derivative, however, is discontinuous at the points α^{-1} and α.

The value $n(1+\sqrt{c})/(1-\sqrt{c})$ is the asymptotics of the largest zero $x_{\max}$ of the Meixner polynomials $p_n(x)$. Note that the inequality $x_{\max} < n(1+\sqrt{c})/(1-\sqrt{c}) + o(n)$ is required by the position of the Gerschgorin circles of the associated Jacobi matrix. For further and more precise inequalities for zeros of $p_n(x)$ see [7]. Some asymptotic results can be found in [8].

By the substitution $\phi = \ln\sqrt{c}$, we can make similarities with the Theorem 1 more transparent. Indeed, $f(z)=(\sinh\phi)/z+\cosh\phi$, $\alpha=-\coth(\phi/2)$. One might have expected such similarities, because the Meixner–Pollaczek polynomials are obtained from the Meixner polynomials by continuing the latter in the variable x and parameter c into the complex plane [14].

The asymptotic equation (25) for the Charlier polynomials can be regarded as a particular case of the one for the Meixner polynomials when $c\to 0$.

In Section 5 we will consider another approach (initiated in [13]) to calculation of the asymptotic density of zeros of orthogonal polynomials. It is based on the study of the recurrence relation rather that the difference equation. This approach allows us to give an alternative derivation of $\rho(z)$ for Meixner and Meixner–Pollaczek polynomials (Theorem 1 and the first part of Theorem 2).

5. Eigenvalue density of Jacobi matrices

Consider a symmetric Jacobi matrix J with real matrix elements ($J_{ik}=J_{ki}$, $i,k=0,1,2,\ldots$, $J_{ik}=0$ if $|i-k|>1$) such that $J_{nn}\sim a\varphi(n)$, $J_{n\ n+1}\sim b\varphi(n)/2$ as $n\to\infty$, where a and b are not simultaneously zero. The function $\varphi(x):\mathbf{R}^+\to\mathbf{R}^+$ is nondecreasing and such that $\lim_{n\to\infty}\varphi(n+x)/\varphi(n)=1$ for any $x\in\mathbf{R}$; also let $\lim_{n\to\infty}\varphi(nt)/\varphi(n)=\psi(t)$ exist for $t\in[0,1]$, and the function $g(t)=\mathrm{d}t/\mathrm{d}\psi$ be continuous. We will call the class of such matrices $A_{\varphi(n)}$.

Let $J(n)$ be the $(n+1)\times(n+1)$ truncated J: $J(n)=\|J_{ik}\|_{i,k=0}^{n}$. It follows from [13,15] that for $k=0,1,\ldots$ [4]

$$\mu_k \equiv \lim_{n\to\infty}\frac{1}{n}\operatorname{Tr}\left[\frac{J(n)}{\varphi(n)}\right]^k = \frac{1}{\pi}\int_0^1 \psi^k(t)\,\mathrm{d}t\int_{a-|b|}^{a+|b|}\frac{x^k\,\mathrm{d}x}{\sqrt{b^2-(x-a)^2}}. \tag{35}$$

Suppose there exists a solution $\rho(z)$ to the moment problem $\mu_k=\int_{-\infty}^{\infty} z^k\rho(z)\,\mathrm{d}z$, $k=0,1,\ldots$. Since a and b are finite, the function $\rho(z)\equiv\rho(z,a,b)$ has a bounded support which implies that the moment problem is determinate, and hence, μ_k, $k=0,1,\ldots$ uniquely define $\rho(z,a,b)$. The function $\rho(z,a,b)$ is the density of eigenvalues of $J(n)/\varphi(n)$ in the limit $n\to\infty$. So up to a uniform scaling of the spectrum, $\rho(z,a,b)$ depends only on a/b.

On the other hand, recall that the orthogonal polynomials satisfy the recurrence relation [6]:

$$x p_n(x) = \alpha_n p_{n+1} + \beta_n p_n + \gamma_n p_{n-1}, \quad n=0,1,\ldots,\ \gamma_0 p_{-1}=0. \tag{36}$$

[4] The values μ_k for $b=0$ are obtained from (35) using continuity in b.

After the transformation $q_0 = p_0$, $q_n(x) = \sqrt{(\alpha_0\alpha_1\cdots\alpha_{n-1})/(\gamma_1\gamma_1\cdots\gamma_n)}\,p_n(x)$, $n = 1, 2, \ldots$, it takes the form

$$xq_n(x) = J_{n\ n+1}q_{n+1} + J_{nn}q_n + J_{n\ n-1}q_{n-1}, \qquad n = 0, 1, \ldots,$$

$$J_{nn} = \beta_n, \quad J_{n\ n+1} = J_{n+1\ n} = \sqrt{\alpha_n\gamma_{n+1}}, \quad J_{0\ -1}q_{-1} = 0, \tag{37}$$

where $J = \|J_{ik}\|_{i,k=0}^{\infty}$ is a symmetric Jacobi matrix. As is well known (e.g., [6]), the zeros of $p_{N+1}(x)$ are eigenvalues of $J(N) = \|J_{ik}\|_{i,k=0}^{N}$. Thus, the asymptotics of the zeros of $p_n(x)$ gives the asymptotics of the eigenvalues of $J(n)$. Notice, that if $J \in A_{\varphi(n)}$ and $\rho(z,a,b)$ is defined as above, we get $\rho(z,a,b) = \rho(z,a,-b)$ by the substitution $\tilde{q}_n(x) = (-1)^n q_n(x)$.

Now let us consider the symmetric Jacobi matrices associated with the Meixner and Meixner–Pollaczek polynomials. For Meixner–Pollaczek [9] we get

$$J_{n\ n+1} = \frac{\sqrt{(n+1)(2\lambda+n)}}{2\sin\phi} \sim \frac{n}{2\sin\phi}; \quad J_{nn} = -(n+\lambda)\cot\phi \sim -n\cot\phi. \tag{38}$$

Thus, $J \in A_n$ and $a/b = -\cos\phi$, $\phi \in (0,\pi)$. So $-1 < a/b < 1$.

For the Meixner polynomials ($\phi = \ln\sqrt{c}$)

$$J_{n\ n+1} = \frac{\sqrt{(n+1)(n+\beta)c}}{c-1} \sim \frac{n}{2\sinh\phi}; \quad J_{nn} = \frac{n+(n+\beta)c}{1-c} \sim -n\coth\phi. \tag{39}$$

Thus, $J \in A_n$ and $a/b = -\cosh\phi$, $\phi \in (-\infty, 0)$. So $-\infty < a/b < -1$.

Now it is easy to verify that, up to a sign, an arbitrary matrix J from A_n for which $|a| \neq |b|$ has the form rJ_0, where $r = \sqrt{|a^2 - b^2|}$ and J_0 is a matrix associated with either Meixner–Pollaczek or Meixner polynomials defined up to a sign. Obviously, $\rho(z,a,b) = \rho_0(z/r, a/r, b/r)/r$, where ρ and ρ_0 are the eigenvalue densities of $J(n)/n$ and $J_0(n)/n$ as $n \to \infty$, respectively.

Now we can reformulate Theorem 1 and the first part of Theorem 2.

Theorem 3. *Let $J \in A_n$, $J_{nn} \sim an$, $J_{n\ n+1} \sim bn/2$ as $n \to \infty$. Then the eigenvalue density of $J(n)/n$ in the limit $n \to \infty$ is the following*:

(1) $0 \leqslant a/b \leqslant 1$,

$$\rho(z,a,b) = \begin{cases} \dfrac{1}{\pi r}\operatorname{arccosh}\left|\dfrac{r^2}{zb} + \dfrac{a}{b}\right|, & z \in [a-b, a+b], \\ 0, & z \notin [a-b, a+b], \end{cases}$$

(2) $a/b > 1$,

$$\rho(z,a,b) = \begin{cases} 1/r, & z \in [0, a-b], \\ \dfrac{1}{\pi r}\arccos\left(-\dfrac{r^2}{zb} + \dfrac{a}{b}\right), & z \in [a-b, a+b], \\ 0, & z \notin [0, a+b], \end{cases}$$

where $0 \leqslant \arccos(x) \leqslant \pi$, $r = \sqrt{|a^2 - b^2|}$.

In the case $a/b > 1$, more precise information about the zeros in the region where $\rho(z) = 1/r$ is given by the function $\chi(z)$ (Theorem 2).

It was sufficient to consider only the case $a \geqslant 0$, $b \geqslant 0$ (substitution $b \to -b$ does not affect the spectrum, and $a \to -a$ inverts it with respect to zero). In what follows, we always assume a and b nonnegative.

For $a = 0$ our result coincides with a particular case of the Nevai–Dehesa–Ullman distribution. This distribution is the following [15]:

$$\rho(z,0,b) = \frac{1}{\pi}\int_{|z/b|}^{1} \frac{g(\omega)\,\mathrm{d}\omega}{\sqrt{b^2\omega^2 - z^2}}, \quad z \in [-b, b]. \tag{40}$$

For $\varphi(n) = n^\gamma$, $\gamma > 0$, we have $g(\omega) = \omega^{-1+1/\gamma}/\gamma$.[5]

It is not difficult to generalize (40) to arbitrary a following the approach of [13,15]. It is, however, interesting because of qualitatively new features which appear when $a > b$. Note that this generalization appears in [16], where a different approach to its derivation is used.

Theorem 4. *Let $J \in A_{\varphi(n)}$, $J_{nn} \sim a\varphi(n)$, $J_{n\,n+1} \sim b\varphi(n)/2$ as $n \to \infty$. Then the eigenvalue density of $J(n)/\varphi(n)$ in the limit $n \to \infty$ is the following*:

(1) $0 \leqslant a/b \leqslant 1$,

$$\rho(z) = \frac{1}{\pi}\int_{z/(a+b\,\mathrm{sign}\,z)}^{1} \frac{g(\omega)\,\mathrm{d}\omega}{\sqrt{b^2\omega^2 - (z-a\omega)^2}} \text{ if } z \in [a-b, a+b], \text{ and } \rho(z) = 0 \text{ otherwise;}$$

(2) $a/b > 1$,

$$\rho(z) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_{(a-b)/(a+b)}^{1} \frac{g(\omega\frac{z}{a-b})\,\mathrm{d}\omega}{\sqrt{b^2\omega^2 - (a-b-a\omega)^2}}, & z \in [0, a-b], \\ \dfrac{1}{\pi}\displaystyle\int_{z/(a+b)}^{1} \frac{g(\omega)\,\mathrm{d}\omega}{\sqrt{b^2\omega^2 - (z-a\omega)^2}}, & z \in [a-b, a+b], \\ 0, & z \notin [0, a+b]. \end{cases}$$

Note that for $\varphi(n) = n^\gamma$, $\gamma > 0$ we can write case 2 in a simpler form

$$\rho(z) = \begin{cases} h(a-b)\left(\dfrac{z}{a-b}\right)^{-1+1/\gamma} & z \in [0, a-b], \\ h(z) & z \in [a-b, a+b], \\ 0 & z \notin [0, a+b], \end{cases}$$

where

$$h(z) = \frac{1}{\pi\gamma}\int_{z/(a+b)}^{1} \frac{\omega^{-1+1/\gamma}\,\mathrm{d}\omega}{\sqrt{b^2\omega^2 - (z-a\omega)^2}}.$$

[5] The results for $\gamma = 0$ are obtained using continuity.

Proof. Analogous to that given by Ullman in the case $a=0$. Rewrite (35) in the form

$$\mu_k = \frac{1}{\pi}\int_0^1 g(\psi)\,\mathrm{d}\psi \int_{a-b}^{a+b} \frac{(\psi x)^k\,\mathrm{d}x}{\sqrt{b^2-(x-a)^2}}. \tag{41}$$

Consider case 1. Changing the variables $z=\psi x$, $\omega=\psi$ in the double integral, we have

$$\mu_k = \frac{1}{\pi}\int_{a-b}^{a+b} z^k\,\mathrm{d}z \int_{z/(a+b\,\mathrm{sign}\,z)}^{1} \frac{g(\omega)\,\mathrm{d}\omega}{\sqrt{b^2\omega^2-(z-a\omega)^2}}, \tag{42}$$

which proves case 1. Similarly, we prove case 2. □

Theorem 1 and the first part of Theorem 2 (that is Theorem 3) are corollaries of Theorem 4 when $\varphi(n)=n$.

Finally, note an important and, in its generality, very difficult problem: what weights of orthogonal polynomials lead to the matrices in $A_{\varphi(n)}$? A review of related results is given in [12]. In this sense, the case $|a|>|b|$ is studied less than the case $|a|<|b|$.

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References

[1] S. Ahmed, M. Bruschi, F. Calogero, M.A. Olshanetsky, A.M. Perelomov, Properties of the zeros of the classical polynomials and of the Bessel functions, Nuovo Cimento 49B (1979) 173–199.

[2] S. Ahmed, M.E. Muldoon, Reciprocal power sums of differences of zeros of special functions, SIAM J. Math. Anal. 14 (1983) 372–382.

[3] F. Calogero, A.M. Perelomov, Asymptotic density of zeros of Hermite polynomials of diverging order and related properties of certain singular integral operators, Lett. Nuovo Cimento 23 (1978) 650–652.

[4] K.M. Case, Sum rules for zeros of polynomials I, II, J. Math. Phys. 21 (1980) 702–708, 709–714.

[5] Y. Chen, M.E.H. Ismail, Asymptotics of extreme zeros of the Meixner–Pollaczek polynomials, J. Comput. Appl. Math. 82 (1997) 59–78.

[6] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[7] M.E.H. Ismail, M.E. Muldoon, A discrete approach to monotonicity of zeros of orthogonal polynomials, Trans. Amer. Math. Soc. 323 (1991) 65–78.

[8] X.-S. Jin, R. Wong, Asymptotic formula for the zeros of the Meixner polynomials, J. Approx. Theory 96 (1999) 281–300.

[9] R. Koekoek, R.F. Swarttouw, The Askey-Scheme of Hypergeometric Orthogonal Polynomials and its q-Analogue, Delft University of Technology, Netherland, 1996.

[10] I.V. Krasovsky, Bethe ansatz for the Harper equation: solution for a small commensurability parameter Phys. Rev. B 59 (1999) 322–328.

[11] L.D. Landau, E.M. Lifshitz, Course of Theoretical Physics, Vol. 3, Quantum Mechanics, Nauka, Moscow, 1989.

[12] D.S. Lubinsky, A survey of general orthogonal polynomials for weights on finite and infinite intervals, Acta Appl. Math. 10 (1987) 237–296;
D.S. Lubinsky, An update on orthogonal polynomials and weighted approximation on the real line, Acta Appl. Math. 33 (1993) 121–164.

[13] P.G. Nevai, J.S. Dehesa, On asymptotic average properties of zeros of orthogonal polynomials, SIAM J. Math. Anal. 10 (1979) 1184–1192.

[14] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer Series in Computational Physics, Springer, Berlin, 1991; Nauka, Moscow, 1985 (also in Russian).

[15] J.L. Ullman, On orthogonal polynomials associated with the infinite interval, in: E.W. Cheney (Ed.), Approximation Theory III, Academic Press, New York, 1980, pp. 889–895.

[16] W. Van Assche, Asymptotics for orthogonal polynomials, Lecture Notes in Mathematics, Vol. 1265, Springer, Berlin, 1987.

[17] W. Van Assche, Norm behavior and zero distribution for orthogonal polynomials with nonsymmetric weights, Constr. Approx. 5 (1989) 329–345.

[18] P.B. Wiegmann, A.V. Zabrodin, Algebraization of difference eigenvalue equations related to $U_q(sl(2))$, Nucl. Phys. B 451 (1995) 699–724.